数理解析研究所講究録 1388 巻 2004 年 1-22

Hardy spaces and generalized fractional integrals

大阪教育大学 教育学部　中井 英一 (Eiichi Nakai)
Department of Mathematics
Osaka Kyoiku University

1. Introduction

The fractional integral I_α $(0<\alpha<n)$ is defined by

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

This is also called the Riesz potential. It is known that

Theorem 1.1 (Hardy-Littlewood-Sobolev). *Let*

$$1<p<q<\infty,\quad -n/p+\alpha=-n/q$$

Then

$$I_\alpha : L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)\quad bdd.$$

This boundedness extended to $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ as Figure 1.

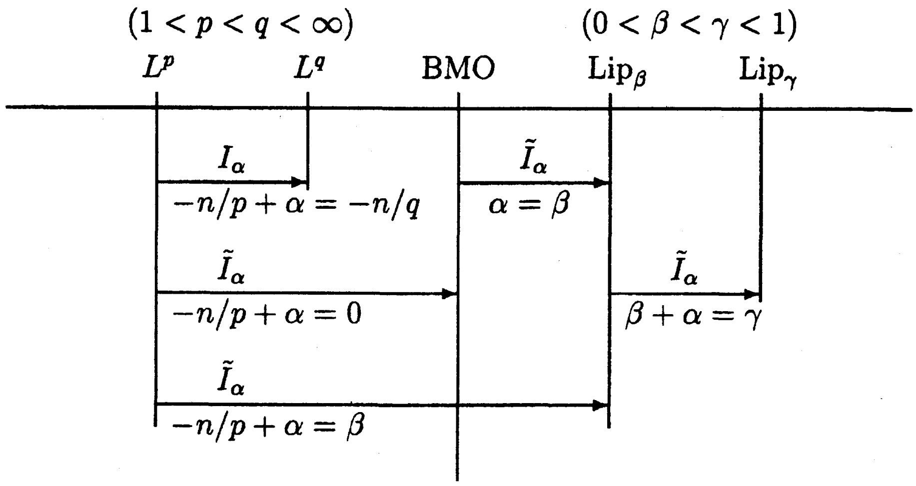

FIGURE 1. Boundedness of fractional integrals

For Hardy spaces, it is also known that the fractional integral is a continuous operator from $H^p(\mathbb{R}^n)$ to $H^q(\mathbb{R}^n)$ (see Figure 2).

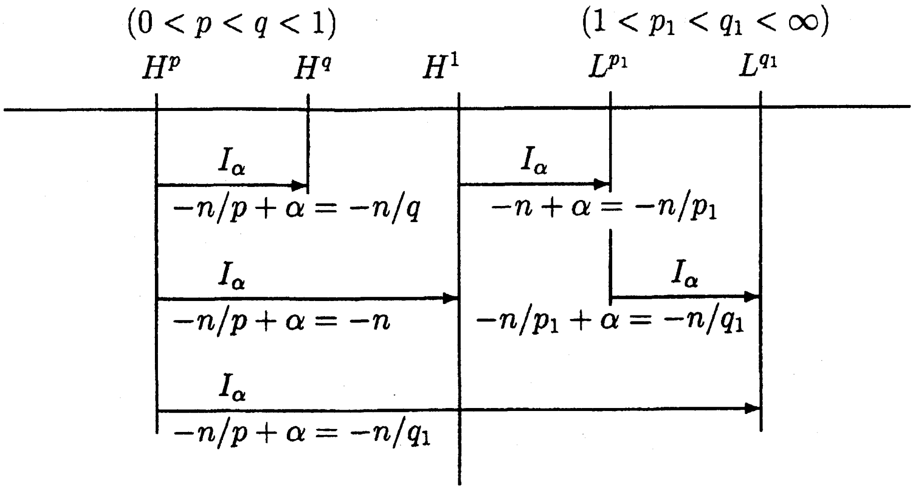

FIGURE 2. Boundedness of fractional integrals

For a function $\rho : (0, +\infty) \to (0, +\infty)$, let

$$I_\rho f(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy.$$

We consider the following conditions on ρ:

$$\int_0^1 \frac{\rho(t)}{t} \, dt < +\infty, \tag{1.1}$$

$$\frac{1}{C} \le \frac{\rho(s)}{\rho(r)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2. \tag{1.2}$$

If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the fractional integral denoted by I_α.

Using I_ρ, the author extended the Hardy-Littlewood-Sobolev theorem to Orlicz spaces and Morrey-Campanato spaces with general growth functions.

In this article, I give a generalization of the Hardy space, and extend the $H^p - H^q$ continuity of I_α.

2. ORLICZ AND MORREY-CAMPANATO SPACES

For functions $\theta, \kappa : (0, +\infty) \to (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r) \quad \text{for} \quad r > 0.$$

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(r) \le C\theta(s) \quad (\theta(r) \ge C\theta(s)) \quad \text{for} \quad r \le s.$$

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\theta(r)}{\theta(s)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2.$$

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi : [0, +\infty) \to [0, +\infty)$. Then $\Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = +\infty$ for $\Phi \in \mathcal{F}$. Let $\Phi(+\infty) = +\infty$.

2.1. **Orlicz space.** For a convex function $\Phi \in \mathcal{F}$, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\, dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\},$$

$$L^\Phi_{weak}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_{r>0} \Phi(r)\, m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi,weak} = \inf \left\{ \lambda > 0 : \sup_{r>0} \Phi(r)\, m\left(r, \frac{f}{\lambda}\right) \le 1 \right\},$$

$$\text{where} \quad m(r, f) = |\{x \in \mathbb{R}^n : |f(x)| > r\}|.$$

Then

$$L^\Phi(\mathbb{R}^n) \subset L^\Phi_{weak}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{\Phi,weak} \le \|f\|_\Phi.$$

$\|f\|_\Phi$ is a norm and $L^\Phi(\mathbb{R}^n)$ is a Banach space. $\|f\|_{\Phi,weak}$ is a quasi-norm and $L^\Phi_{weak}(\mathbb{R}^n)$ is a complete quasi-normed space.

For a function Φ, the complementary function is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \ge 0\}, \quad r \ge 0.$$

For example,

$$\Phi(r) = r^p \quad \Rightarrow \quad L^\Phi = L^p,$$
$$\widetilde{\Phi}(r) \sim r^{p'} \quad \Rightarrow \quad L^{\widetilde{\Phi}} = L^{p'},$$

for $1 < p < \infty$, $1/p + 1/p' = 1$.

$$\Phi(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r, \\ \exp(r^p) & \text{for large } r, \end{cases} \quad \Rightarrow \quad L^\Phi = \exp L^p,$$

$$\widetilde{\Phi}(r) \sim \begin{cases} r(\log(1/r))^{-1/p} & \text{for small } r, \\ r(\log r)^{1/p} & \text{for large } r, \end{cases} \quad \Rightarrow \quad L^{\widetilde{\Phi}} = L(\log L)^{1/p},$$

for $0 < p < \infty$.

A function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

If $1 < p < \infty$, then $\Phi(r) = r^p \in \nabla_2$. For $0 < p < \infty$,

$$\Phi(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r, \\ \exp(r^p) & \text{for large } r, \end{cases}$$

satisfies the ∇_2 condition.

2.2. **Morrey space.** For $1 \le p < \infty$ and a function $\phi : (0, +\infty) \to (0, +\infty)$, let

$$\|f\|_{L_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(x)|^p \, dx \right)^{1/p},$$
$$L_{p,\phi}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L_{p,\phi}} < +\infty \right\}.$$

We assume that ϕ satisfies the doubling condition and that $\phi(r)r^{n/p}$ is almost increasing. If $\phi(r) = r^{(\lambda-n)/p}$ $(0 \le \lambda \le n)$, then $L_{p,\phi}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ which is the classical Morrey space. If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

If $\phi(r) \to 0$ as $r \to 0$, then $L_{p,\phi}(\mathbb{R}^n) = \{0\}$.

2.3. **Campanato space.** For $1 \le p < \infty$ and a function $\phi : (0, +\infty) \to (0, +\infty)$, let

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right)^{1/p},$$
$$\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}_{p,\phi}} < +\infty \right\},$$
$$\text{where} \qquad f_B = \frac{1}{|B|} \int_B f(x) \, dx.$$

We assume that ϕ satisfies the doubling condition and that $\phi(r)r^{n/p}$ is almost increasing. If $\phi(r) = r^{(\lambda-n)/p}$ $(0 \le \lambda \le n+1)$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ which is the classical Campanato space.

If ϕ is almost increasing, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ for all $p > 1$. We denote $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ by $\mathrm{BMO}_\phi(\mathbb{R}^n)$. If $\phi \equiv 1$, then $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$, $0 < \alpha \le 1$, then it is known that $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

If $\phi(r)/r \to 0$ as $r \to 0$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \{0\}$.

3. Boundedness of I_ρ (known results)

In this section, we consider spaces L^Φ, $L_{1,\phi}$ and $\mathcal{L}_{1,\phi}$. So we assume that $\Phi, \Psi \in \mathcal{F}$ are convex, that ϕ and ψ satisfy the doubling condition, that $\phi(r)r^n$ and $\psi(r)r^n$ are almost increasing, and that

$$\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty,$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2.$$

Theorem 3.1 (N [3]). *Let*

$$\frac{\rho(r)}{r^n} \le C\frac{\rho(s)}{s^n} \quad \textit{for} \quad s \le r.$$

If

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\,\Psi^{-1}\left(\frac{1}{r^n}\right), \quad r > 0, \tag{3.1}$$

$$\int_r^{+\infty} \tilde{\Phi}\left(\frac{\rho(t)}{C\int_0^r (\rho(s)/s)\,ds\,\Phi^{-1}(1/r^n)t^n}\right) t^{n-1}\,dt \le C, \quad r > 0,$$

then

$$I_\rho : L^\Phi(\mathbb{R}^n) \to L^\Psi_{weak}(\mathbb{R}^n) \quad \textit{bdd.}$$

Moreover, if $\Phi \in \nabla_2$, *then*

$$I_\rho : L^\Phi(\mathbb{R}^n) \to L^\Psi(\mathbb{R}^n) \quad \textit{bdd.}$$

In this theorem, if $\Phi(r) = r^p$, $\Psi(r) = r^q$, $\rho(r) = r^\alpha$, then (3.1) is equivalent to $-n/p + \alpha = -n/q$. Actually,

$$\Phi^{-1}\left(\frac{1}{r^n}\right) = r^{-n/p},$$

$$\int_0^r \frac{\rho(t)}{t}\,dt = \frac{r^\alpha}{\alpha},$$

$$\Psi^{-1}\left(\frac{1}{r^n}\right) = r^{-n/q},$$

and

$$r^{-n/p}r^\alpha \le Cr^{-n/q} \quad \text{for all} \quad r > 0 \quad \Leftrightarrow \quad -n/p + \alpha = -n/q.$$

Example 3.1. Let ρ_α satisfy the doubling condition and

$$\rho_\alpha(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\ (\log r)^{\alpha-1} & \text{for large } r, \end{cases} \quad \alpha > 0. \tag{3.2}$$

Then

$$\int_0^r \frac{\rho_\alpha(t)}{t}\,dt \sim \begin{cases} 1/(\log(1/r))^\alpha & \text{for small } r, \\ (\log r)^\alpha & \text{for large } r. \end{cases}$$

For $0 < p < 1/\alpha$, $1/q = 1/p - \alpha$, we have

$$I_{\rho_\alpha} : \exp L^p(\mathbb{R}^n) \to \exp L^q(\mathbb{R}^n) \quad \textit{bdd.}$$

We define the modified version of I_ρ as follows:

$$\tilde{I}_\rho f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right) dy.$$

Theorem 3.2 (N [5]). *Let*

$$\frac{\rho(r)}{r^{n+1}} \le C \frac{\rho(s)}{s^{n+1}} \quad \textit{for} \quad s \le r,$$
$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \le C|r-s| \frac{\rho(r)}{r^{n+1}} \quad \textit{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2.$$

If

$$\phi(r) \int_0^r \frac{\rho(t)}{t}\,dt \le C\psi(r),$$
$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\,dt \le C\frac{\psi(r)}{r},$$

then

$$\tilde{I}_\rho : L_{1,\phi}(\mathbb{R}^n) \to \mathcal{L}_{1,\psi}(\mathbb{R}^n) \quad \textit{bdd.}$$

We have the following relation between L^Φ and $L_{1,\phi}$:

Theorem 3.3 (N [5]). *Let* $\phi(r) = \Phi^{-1}(1/r^n)$. *Then*

$$L^\Phi(\mathbb{R}^n) \subset L^{1,\phi}(\mathbb{R}^n), \quad \textit{and} \quad \|f\|_{L^{1,\phi}} \le C\|f\|_\Phi. \tag{3.3}$$

Moreover, if $\Phi \in \nabla_2$, *then*

$$L^\Phi_{weak}(\mathbb{R}^n) \subset L^{1,\phi}(\mathbb{R}^n), \quad \textit{and} \quad \|f\|_{L^{1,\phi}} \le C\|f\|_{\Phi,weak}. \tag{3.4}$$

Combining Theorems 3.2 and 3.3, we have the following:

Corollary 3.4 (N [5]). *Let*

$$\frac{\rho(r)}{r^{n+1}} \le C \frac{\rho(s)}{s^{n+1}} \quad \textit{for} \quad s \le r,$$
$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \le C|r-s| \frac{\rho(r)}{r^{n+1}} \quad \textit{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2,$$

and ψ be almost increasing. If

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\psi(r),$$
$$\int_r^{+\infty} \frac{\rho(t)\Phi^{-1}(1/t^n)}{t^2}\,dt \le C\frac{\psi(r)}{r},$$

then

$$\tilde{I}_\rho : L^\Phi(\mathbb{R}^n) \to \mathrm{BMO}_\psi(\mathbb{R}^n) \quad \textit{bdd.}$$

Theorem 3.5 (N [5]). *Let*

$$\int_r^{+\infty} \frac{\rho(t)}{t^2}\,dt \le C\frac{\rho(r)}{r}, \tag{3.5}$$

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le C|r-s|\frac{\rho(r)}{r^{n+1}} \quad \textit{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2. \tag{3.6}$$

If

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt \le C\psi(r),$$
$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\,dt \le C\frac{\psi(r)}{r},$$

then

$$\tilde{I}_\rho : \mathcal{L}_{1,\phi}(\mathbb{R}^n) \to \mathcal{L}_{1,\psi}(\mathbb{R}^n) \quad \textit{bdd.}$$

Remark 3.1. Since $\tilde{I}_\rho 1$ is a constant, $\tilde{I}_\rho$ is well defined as an operator from $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ to $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$.

Corollary 3.6 (N [3]). *Let*

$$\int_r^{+\infty} \frac{\rho(t)}{t^2}\,dt \le C\frac{\rho(r)}{r}, \tag{3.7}$$

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le C|r-s|\frac{\rho(r)}{r^{n+1}} \quad \textit{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{3.8}$$

and, ϕ and ψ be almost increasing. If

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt \le C\psi(r),$$
$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\,dt \le C\frac{\psi(r)}{r},$$

then

$$\tilde{I}_\rho : \mathrm{BMO}_\phi(\mathbb{R}^n) \to \mathrm{BMO}_\psi(\mathbb{R}^n) \quad \textit{bdd.}$$

Example 3.2. Let

$$\rho_\alpha(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\ (\log r)^{\alpha-1} & \text{for large } r, \end{cases} \quad \alpha > 0, \tag{3.9}$$

$$\phi_\beta(r) = \begin{cases} (\log(1/r))^{-\beta} & \text{for small } r, \\ (\log r)^{\beta} & \text{for large } r. \end{cases} \tag{3.10}$$

Let

$$\Phi(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r, \\ \exp(r^p) & \text{for large } r, \end{cases} \quad p > 0.$$

Then

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \sim \begin{cases} 1/(\log(1/r))^{1/p} & \text{for small } r, \\ (\log r)^{-1/p} & \text{for large } r. \end{cases}$$

Hence we have

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \int_0^r \frac{\rho_\alpha(t)}{t}\, dt \sim \phi_{-1/p+\alpha},$$

$$\phi_\beta(r) \int_0^r \frac{\rho_\alpha(t)}{t}\, dt \sim \phi_{\alpha+\beta}(r).$$

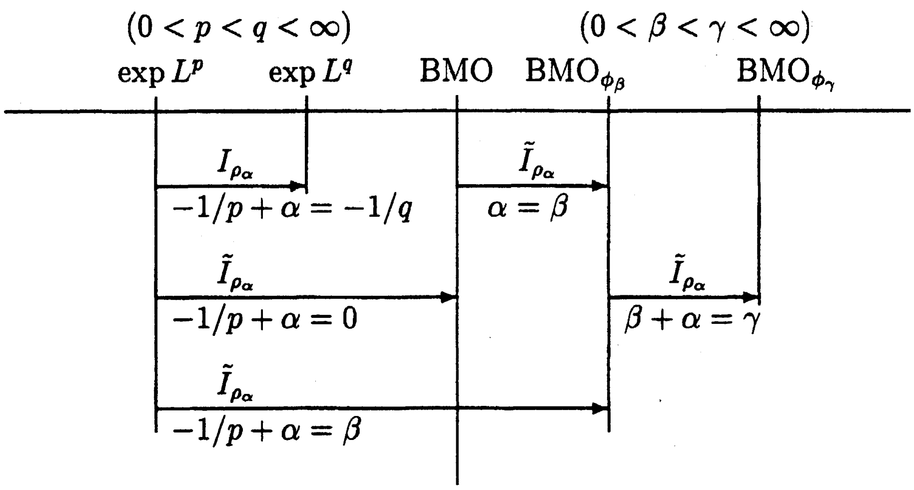

FIGURE 3. Boundedness of generalized fractional integrals

4. Hardy space defined by generalized atoms

Definition 4.1. Let $\Phi \in \mathcal{F}$, $1 < q \le +\infty$ and $r^{1/q}\Phi^{-1}(1/r)$ be almost decreasing. A function a on $\mathbb{R}^n$ is called a (Φ, q)-atom if there exists a ball B such that

$$\begin{cases} (i) & \operatorname{supp} a \subset B, \\ (ii) & \|a\|_q \le |B|^{1/q}\Phi^{-1}\left(\dfrac{1}{|B|}\right), \\ (iii) & \displaystyle\int a(x)\,dx = 0. \end{cases}$$

We denote by $A(\Phi, q)$ the set of all (Φ, q)-atoms.

A function a on $\mathbb{R}^n$ is called a (Φ, q)-block if there exists a ball B such that (i) and (ii) hold. We denote by $B(\Phi, q)$ the set of all (Φ, q)-blocks.

Definition 4.2. Let $\Phi \in \mathcal{F}$, $1 < q \le +\infty$, $r^{1/q}\Phi^{-1}(1/r)$ be almost decreasing, $U \in \mathcal{F}$ and U be concave. We define a space $H_U^{\Phi,q} = H_U^{\Phi,q}(\mathbb{R}^n) \subset \mathfrak{D}'$ as follows:

$f \in H_U^{\Phi,q}(\mathbb{R}^n)$ if and only if there exist sequances $\{a_j\} \subset A(\Phi, q)$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j a_j \text{ in } \mathfrak{D}' \quad \text{and} \quad \sum_j U(\lambda_j) < +\infty. \tag{4.1}$$

In general, the expression (4.1) is not unique. We define

$$\|f\|_{H_U^{\Phi,q}} = \inf\left\{ U^{-1}\left(\sum_j U(\lambda_j)\right) : f = \sum_j \lambda_j a_j \text{ in } \mathfrak{D}' \right\},$$

where the infimum is taken over all expressions (4.1).

We also define a space $B_U^{\Phi,q} = B_U^{\Phi,q}(\mathbb{R}^n) \subset \mathfrak{D}'$ by using (Φ, q)-blocks instead of (Φ, q)-atoms.

If $U \in \mathcal{F}$ is concave, then

$$cU(r) \le U(cr) \quad \text{for} \quad 0 \le c \le 1.$$

Hence, for positive numbers r and s,

$$U(r+s) = \frac{r}{r+s}U(r+s) + \frac{s}{r+s}U(r+s) \le U(r) + U(s).$$

So we have

$$\sum_j \lambda_j \le U^{-1}\left(\sum_j U(\lambda_j)\right).$$

$H_U^{\Phi,q}(\mathbb{R}^n)$ is a linear space. Let $d(f, g) = U(\|f - g\|_{H_U^{\Phi,q}})$ for $f, g \in H_U^{\Phi,q}(\mathbb{R}^n)$. Then $d(f, g)$ is a metric and $H_U^{\Phi,q}$ is complete with respect to this metric. Let

$I(r) = r$. Then $\|f\|_{H_I^{\Phi,q}}$ is a norm and $H_I^{\Phi,q}$ is a Banach space. We have similar properties for $B_U^{\Phi,q}$.

For $q = \infty$, we denote $H_U^{\Phi,q} = H_U^{\Phi}$.
For $\Phi(r) = 1/U(1/r)$, we denote $H_U^{\Phi,q} = H^{\Phi,q}$.
For $q = \infty$ and $\Phi(r) = 1/U(1/r)$, we denote $H_U^{\Phi,q} = H^{\Phi}$.
If $\Phi(r) = r^p$, $n/(n+1) < p \le 1$, then $H^{\Phi} = H^p$.
We have

$$\begin{aligned}
1 < q_1 < q_2 \le \infty \quad &\Rightarrow \quad H_U^{\Phi,q_2}(\mathbb{R}^n) \subset H_U^{\Phi,q_1}(\mathbb{R}^n), \\
\Psi(r) \le \Phi(Cr) \quad \text{for all} \quad r > 0 \quad &\Rightarrow \quad H_U^{\Phi,q}(\mathbb{R}^n) \subset H_U^{\Psi,q}(\mathbb{R}^n), \\
V(r) \le CU(r) \quad \text{for} \quad 0 \le r \le 1 \quad &\Rightarrow \quad H_U^{\Phi,q}(\mathbb{R}^n) \subset H_V^{\Phi,q}(\mathbb{R}^n), \\
\text{for all concave function } U \in \mathcal{F}, \quad &H_U^{\Phi,q}(\mathbb{R}^n) \subset H_I^{\Phi,q}(\mathbb{R}^n),
\end{aligned}$$

where the inclusion mapping are continuous.

For $1 < q \le \infty$, L^q_{comp} is dense in $B_U^{\Phi,q}$. Let

$$L^{q,0}_{\text{comp}}(\mathbb{R}^n) = \left\{ f \in L^q_{\text{comp}}(\mathbb{R}^n) : \int f(x)\,dx = 0 \right\}.$$

Then $L^{q,0}_{\text{comp}}$ is dense in $H_U^{\Phi,q}$.

Theorem 4.1. *Let $1 < q \le \infty$, $1/q + 1/q' = 1$, $\Phi \in \mathcal{F}$, Φ^{-1} satisfy the doubling condition, $r^{1/q}\Phi^{-1}(1/r)$ be almost decreasing, $U \in \mathcal{F}$ and U be concave. Assume that*

$$\sup_{0<s<1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0).$$

If

$$\phi(r) = \frac{1}{r^n \Phi^{-1}\left(\frac{1}{r^n}\right)},$$

then

$$\left(B_U^{\Phi,q}(\mathbb{R}^n)\right)^* = L_{q',\phi}(\mathbb{R}^n).$$

If $\Phi(r)/r \to 0$ as $r \to +\infty$, then $\phi(r) \to 0$ as $r \to 0$. Hence

$$\left(B_U^{\Phi,q}(\mathbb{R}^n)\right)^* = \{0\}.$$

Remark 4.1. For $B = B(z,r)$,

$$\phi(r) \sim \frac{1}{|B|\Phi^{-1}\left(\frac{1}{|B|}\right)}.$$

Theorem 4.2. *Let* $1 < q \le \infty$, $1/q + 1/q' = 1$, $\Phi \in \mathcal{F}$, $r^{1/q}\Phi^{-1}(1/r)$ *be almost decreasing,* $U \in \mathcal{F}$ *and* U *be concave. Assume that*

$$\sup_{0<s<1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0).$$

If

$$\phi(r) = \frac{1}{r^n \Phi^{-1}\left(\frac{1}{r^n}\right)},$$

then

$$\left(H_U^{\Phi,q}(\mathbb{R}^n)\right)^* = \mathcal{L}_{q',\phi}(\mathbb{R}^n).$$

If $\Phi(r)/r^{n/(n+1)} \to 0$ as $r \to +\infty$, then $\phi(r)/r \to 0$ as $r \to 0$. Hence

$$\left(H_U^{\Phi,q}(\mathbb{R}^n)\right)^* = \{0\}.$$

Example 4.1. If $\Phi(r) = r$, then $\phi(r) \equiv 1$. In this case, we have

$$\left(H_U^{1,q}(\mathbb{R}^n)\right)^* = \mathrm{BMO}(\mathbb{R}^n).$$

Example 4.2. For $\beta \in \mathbb{R}$, we define a function $\Phi_\beta \in \mathcal{F}$ as follows

$$\Phi_\beta(r) = \begin{cases} r\,(\log(1/r))^{-\beta} & \text{for small } r, \\ r\,(\log r)^{\beta} & \text{for large } r. \end{cases} \tag{4.2}$$

Then Φ_β is concave for $\beta < 0$, and Φ_β is convex for $\beta > 0$. In this case, we have

$$\Phi_\beta{}^{-1}(r) \sim \begin{cases} r\,(\log(1/r))^{\beta} & \text{for small } r, \\ r\,(\log r)^{-\beta} & \text{for large } r. \end{cases}$$

$$\Phi_\beta{}^{-1}\left(\frac{1}{r^n}\right) \sim \begin{cases} r^{-n}\,(\log(1/r))^{-\beta} & \text{for small } r, \\ r^{-n}\,(\log r)^{\beta} & \text{for large } r. \end{cases}$$

Let

$$\phi_\beta(r) = \begin{cases} (\log(1/r))^{-\beta} & \text{for small } r, \\ (\log r)^{\beta} & \text{for large } r. \end{cases} \tag{4.3}$$

Then

$$\phi_{-\beta}(r) \sim \frac{1}{r^n \Phi_\beta{}^{-1}\left(\frac{1}{r^n}\right)}.$$

If $\beta < 0$, then $\phi_{-\beta}$ is almost increasing and

$$\left(H_U^{\Phi_\beta,q}(\mathbb{R}^n)\right)^* = \mathrm{BMO}_{\phi_{-\beta}}(\mathbb{R}^n). \tag{4.4}$$

If $\beta > 0$, then $\phi_{-\beta}$ is almost decreasing and

$$\left(H_U^{\Phi_\beta,q}(\mathbb{R}^n)\right)^* = \mathcal{L}_{q',\phi_{-\beta}}(\mathbb{R}^n).$$

Proposition 4.3. *Let Φ, q, U be as in Definition 4.2. If*

$$\frac{1}{U^{-1}(Cr)} \le \Phi^{-1}\left(\frac{1}{r}\right) \le \frac{U^{-1}\left(\frac{Cs}{r}\right)}{U^{-1}(s)} \quad \textit{for} \quad 0 < s \le r < +\infty,$$
$$U(rs) \le CU(r)U(s) \quad \textit{for} \quad 0 < r, s \le 1,$$

then

$$H_U^{\Phi,q}(\mathbb{R}^n) = H_U^{\Phi,\infty}(\mathbb{R}^n).$$

Example 4.3. Let $n/(n+1) \le p_1 \le p_2 \le 1$ and

$$\Phi(r) = 1/U(1/r) = \begin{cases} r^{p_1} & \text{for small } r, \\ r^{p_2} & \text{for large } r. \end{cases}$$

then

$$H^{\Phi,q}(\mathbb{R}^n) = H^{\Phi,\infty}(\mathbb{R}^n).$$

5. Proofs of Theorem 4.2

To prove Theorem 4.2, we state the following lemma.

Lemma 5.1. *Let*

$$\sup_{0<s<1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0).$$

If $\ell \in \left(H_U^{\Phi,q}(\mathbb{R}^n)\right)^*$, then

$$\|\ell\| = \sup\left\{|\ell(f)| : \|f\|_{H_U^{\Phi,q}} \le 1\right\} < +\infty.$$

Proof of Theorem 4.2. Let $g \in \mathcal{L}_{q',\phi}(\mathbb{R}^n)$. For a (Φ, q)-atom a, $ag \in L^1(\mathbb{R}^n)$ and

$$\int a(x)g(x)\,dx = \int a(x)(g(x) - g_B)\,dx,$$

where $\operatorname{supp} a \subset B = B(z,r)$. Then

$$\begin{aligned}
\left|\int a(x)g(x)\,dx\right| &\le \|a\|_q \left(\int_B |g(x)-g_B|^{q'}\,dx\right)^{1/q'} \\
&\le |B|^{1/q}\Phi^{-1}\left(\frac{1}{|B|}\right)\left(\int_B |g(x)-g_B|^{q'}\,dx\right)^{1/q'} \\
&= |B|\Phi^{-1}\left(\frac{1}{|B|}\right)\left(\frac{1}{|B|}\int_B |g(x)-g_B|^{q'}\,dx\right)^{1/q'} \\
&\sim \frac{1}{\phi(r)}\left(\frac{1}{|B|}\int_B |g(x)-g_B|^{q'}\,dx\right)^{1/q'} \le \|g\|_{\mathcal{L}_{q',\phi}}.
\end{aligned}$$

For $f \in L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n)$, $fg \in L^1(\mathbb{R}^n)$. Let

$$f = \sum_j \lambda_j a_j, \quad U^{-1}\left(\sum_j U(|\lambda_j|)\right) \le 2\|f\|_{H_U^{\Phi,q}}.$$

We can show

$$\int f(x)g(x)\,dx = \sum_j \lambda_j \int a_j(x)g(x)\,dx.$$

Then

$$\begin{aligned}
\left|\int f(x)g(x)\,dx\right| &\le C\left(\sum_j |\lambda_j|\right)\|g\|_{\mathcal{L}_{q',\phi}} \\
&\le CU^{-1}\left(\sum_j U(|\lambda_j|)\right)\|g\|_{\mathcal{L}_{q',\phi}} \le 2C\|f\|_{H_U^{\Phi,q}}\|g\|_{\mathcal{L}_{q',\phi}}.
\end{aligned}$$

Conversely, let $\ell \in \left(H_U^{\Phi,q}(\mathbb{R}^n)\right)^*$. Fix $B = B(z,r)$. For $f \in L^{q,0}(B)$, let

$$a(x) = \begin{cases} |B|^{1/q}\Phi^{-1}\left(\frac{1}{|B|}\right)\|f\|_q^{-1} f(x) & x \in B \\ 0 & x \notin B. \end{cases}$$

then a is a (Φ,q)-atom. Therefore, by Lemma 5.1, we have

$$|\ell(a)| \le \|\ell\|,$$

i.e.

$$\frac{|\ell(f)|}{\|f\|_q} \le \|\ell\|\left(|B|^{1/q}\Phi^{-1}\left(\frac{1}{|B|}\right)\right)^{-1} \sim \|\ell\|\phi(r)|B|^{1/q'}, \quad f \in L^{q,0}(B).$$

Since $L^{q,0}(B)$ is a subspace of $L^q(B)$, by the Hahn-Banach theorem, we have

$$\|\ell\|_{(L^q(B))^*} \le C\|\ell\|\phi(r)|B|^{1/q'}.$$

Using the duality $(L^q)^* = L^{q'}$, we have

$\exists h^B \in L^{q'}(B)$ s.t.

$$\ell(f) = \int_B f(x)h^B(x)\,dx, \quad \|h^B\|_{L^{q'}(B)} \le C\|\ell\|\phi(r)|B|^{1/q'}.$$

Let $g^B(x) = h^B(x) - (h^B)_B$, $x \in B$. Then

$$(g^B)_B = 0, \quad \|g^B\|_{L^{q'}(B)} \le C\|\ell\|\phi(r)|B|^{1/q'}$$

$$\ell(f) = \int_B f(x)h^B(x)\,dx = \int_B f(x)g^B(x)\,dx, \quad f \in L^{q,0}(B).$$

For every ball B, we have g^B as above. For the class $\{g^B\}_B$,

$$\exists g \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{s.t.} \quad \text{for each ball } B, \quad g - g_B = g^B \quad \text{on } B.$$

And we have

$$g \in \mathcal{L}_{q',\phi}(\mathbb{R}^n), \quad \|g\|_{\mathcal{L}_{q',\phi}} \le C\|\ell\|,$$

$$\ell(f) = \int f(x)g(x)\,dx \quad \text{for } f \in L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n). \quad \square$$

6. Continuity of I_ρ on Hardy spaces

In this section, we assume that $\Phi, \Psi, U, V \in \mathcal{F}$, that Φ^{-1} and Ψ^{-1} satisfy the doubling condition, that U and V are concave, that $1 < q \le \infty$, $1/q + 1/q' = 1$, and that

$$\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty,$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2.$$

In Theorems 6.1 and 6.2, let

$$\frac{\rho(r)}{r^{n+1}} \le C\frac{\rho(s)}{s^{n+1}} \quad \text{for} \quad s \le r,$$

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le C|r-s|\frac{\rho(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2.$$

Theorem 6.1. *Let*

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right), \quad r > 0,$$

$$V(rs) \le CV(r)U(s), \quad 0 \le r, s \le 1,$$

and $0 < \exists\theta < 1$ *s.t.*

$$\int_r^{+\infty} V\left(\left(\frac{\Psi^{-1}(1/t^n)}{\Psi^{-1}(1/r^n)}\right)^{(1/\theta)-1}\right) t^{-1}\,dt \le C, \quad r > 0,$$

$$\int_r^{+\infty} t^n \left(\Psi^{-1}\left(\frac{1}{t^n}\right)\right)^{1/\theta} t^{-1}\,dt \le Cr^n \left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{1/\theta}, \quad r > 0,$$

$$\frac{\rho(r)}{r^{n+1}}\left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{-1/\theta} \quad \textit{is almost decreasing.}$$

Let

$$\int_1^{+\infty} \frac{\rho(t)}{t^2}\,dt < +\infty.$$

Then

$$I_\rho : H_U^\Phi(\mathbb{R}^n) \to H_V^\Psi(\mathbb{R}^n) \quad \textit{conti.}$$

From (6.1), we have

$$\forall C_1 > 0\ \exists C_2 > 0 \quad \text{s.t.} \quad 0 < s, t \le C_1 \Rightarrow V(st) \le C_2 V(s) U(t).$$

Theorem 6.2. *Let* Ψ *is convex, and*

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right), \quad r > 0,$$

$$\int_r^{+\infty} \Psi\left(\frac{\rho(t) r^{n+1}\Phi^{-1}(1/r^n)}{t^{n+1}}\right) t^{n-1}\,dt \le C, \quad r > 0.$$

Then

$$I_\rho : H_U^\Phi(\mathbb{R}^n) \to L^\Psi(\mathbb{R}^n) \quad \textit{conti.}$$

Example 6.1. Let

$$\rho_\alpha(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\ (\log r)^{\alpha-1} & \text{for large } r, \end{cases} \quad \alpha > 0, \tag{6.1}$$

$$\Phi_\beta(r) = \begin{cases} r\,(\log(1/r))^{-\beta} & \text{for small } r, \\ r\,(\log r)^{\beta} & \text{for large } r. \end{cases} \tag{6.2}$$

Then

$$\Phi_\beta{}^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho_\alpha(t)}{t}\,dt \sim \Phi_{\beta+\alpha}{}^{-1}\left(\frac{1}{r^n}\right),$$

and the assumptions of Theorems 6.1 and 6.2 are satisfied. So we have the following continuities in Figure 4.

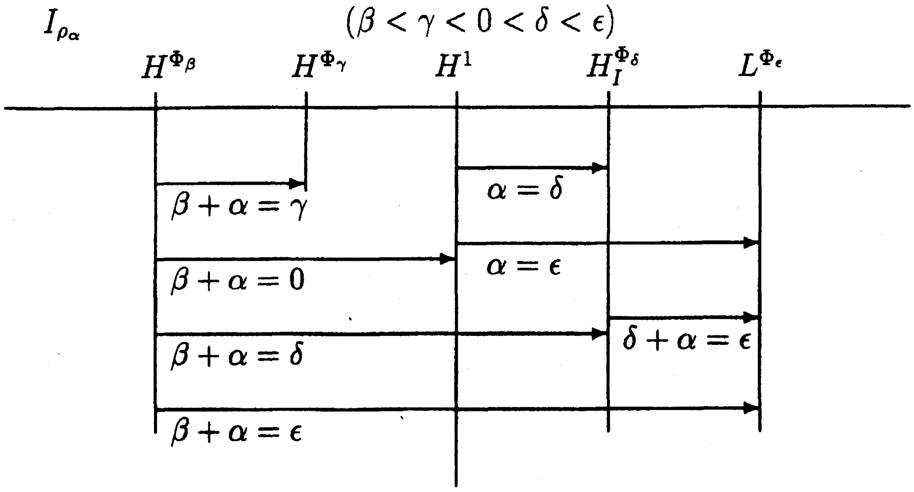

FIGURE 4. Continuity of generalized fractional integrals

In Theorems 6.3 and 6.4, let

$$\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty,$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2,$$

$$\frac{\rho(r)}{r^n} \le C\frac{\rho(s)}{s^n} \quad \text{for} \quad s \le r.$$

Theorem 6.3. *Let*

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right), \qquad r > 0,$$

$$V(rs) \le CV(r)U(s), \quad 0 \le r, s \le 1,$$

and $0 < \exists\theta < 1$ *s.t.*

$$\int_r^{+\infty} V\left(\left(\frac{\Psi^{-1}(1/t^n)}{\Psi^{-1}(1/r^n)}\right)^{(1/\theta)-1}\right) t^{-1}\,dt \le C, \quad r > 0,$$

$$\frac{\rho(r)}{r^n}\left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{-1/\theta} \quad \textit{is almost decreasing.}$$

Then

$$I_\rho : B_U^{\Phi,\infty}(\mathbb{R}^n) \to B_V^{\Psi,\infty}(\mathbb{R}^n) \quad \textit{conti.}$$

Theorem 6.4. *Let Ψ is convex, and*

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right), \quad r>0,$$

$$\int_r^{+\infty} \Psi\left(\frac{\rho(t)r^n\Phi^{-1}(1/r^n)}{t^n}\right) t^{n-1}\,dt \le C, \quad r>0.$$

Then

$$I_\rho : B_U^{\Phi,\infty}(\mathbb{R}^n) \to L^\Psi(\mathbb{R}^n) \quad \textit{conti.}$$

7. Proof of Theorems 6.1–6.4

To prove the theorems, we define molecules and state propositions.

Definition 7.1. Let $\Phi \in \mathcal{F}$, $1 < q \le \infty$, and $0 < \theta < 1$. A function M on $\mathbb{R}^n$ is called a (Φ, q, θ)-molecule if

$$\text{(7.1)} \qquad \begin{cases} (i) & \exists z \in \mathbb{R}^n \ \text{ s.t. } \ \|M\|_q^{1-\theta}\left\|b(|\cdot - z|^n)^{1/\theta}M(\cdot)\right\|_q^\theta < +\infty, \\ (ii) & \int M(x)\,dx = 0, \end{cases}$$

where

$$b(r) = \left(r^{1/q}\Phi^{-1}(1/r)\right)^{-1}.$$

Let

$$\mathcal{N}(M) = \mathcal{N}^{\Phi,q,\theta}(M) = \inf_{z\in\mathbb{R}^n} \|M\|_q^{1-\theta}\left\|b(|\cdot - z|^n)^{1/\theta}M(\cdot)\right\|_q^\theta.$$

Proposition 7.1. *Let*

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right), \quad r>0,$$

$$\text{(7.2)} \qquad 0 < \exists\theta < 1 \quad s.t. \quad \frac{\rho(r)}{r^{n+1}}\left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{-1/\theta} \quad \textit{is almost decreasing}$$

$$\text{(7.3)} \qquad \int_1^{+\infty} \frac{\rho(t)}{t^2}\,dt < +\infty.$$

If $a \in A(\Phi,\infty)$, then $I_\rho a$ is a (Ψ,∞,θ)-molecule and $\mathcal{N}(I_\rho a) \le C$, where C is independent of $a \in A(\Phi,\infty)$.

Remark 7.1. If we omit (7.2) and (7.3), then $I_\rho a$ satisfies (i) in (7.1) for each $a \in B(\Phi,\infty)$, and $\mathcal{N}(I_\rho a) \le C$, where C is independent of $a \in B(\Phi,\infty)$.

Proposition 7.2. *Let $0 < \exists\theta < 1$ s.t.*

$$\int_r^\infty V\left(\left(\frac{\Psi^{-1}(1/t^n)}{\Psi^{-1}(1/r^n)}\right)^{(1/\theta)-1}\right) t^{-1}\,dt \le C, \quad r>0,$$

$$\text{(7.4)} \qquad \int_r^{+\infty} t^n\left(\Psi^{-1}\left(\frac{1}{t^n}\right)\right)^{1/\theta} t^{-1}\,dt \le Cr^n\left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{1/\theta}, \quad r>0.$$

If M is a (Ψ, q, θ)-molecule, then $M \in H_V^{\Psi,q}(\mathbb{R}^n)$, and

$$\forall C_1 > 0 \, \exists C_2 > 0 \quad s.t. \quad \mathcal{N}^{\Psi,q,\theta}(M) \le C_1 \Rightarrow \|M\|_{H_V^{\Psi,q}} \le C_2.$$

Remark 7.2. If we omit (7.4), then we have a similar result for $B_V^{\Phi,q}(\mathbb{R}^n)$.

Proof of Theorem 6.1. Let

$$f \in L^{\infty,0}_{\text{comp}}(\mathbb{R}^n), \quad \|f\|_{H_U^\Phi} \le 1,$$
$$f = \sum_j \lambda_j a_j, \quad \{a_j\} \subset A(\Phi, \infty),$$
$$U^{-1}\left(\sum_j U(\lambda_j)\right) \le 2\|f\|_{H_U^\Phi}.$$

By Proposition 7.1 and Proposition 7.2, we have

$$I_\rho a_j = \sum_k \lambda_{j,k} a_{j,k}, \quad \{a_{j,k}\} \subset A(\Psi, \infty),$$
$$V^{-1}\left(\sum_k V(\lambda_{j,k})\right) \le C \quad \text{independent of } j.$$

We also can show that

$$I_\rho f = \sum_j \lambda_j I_\rho a_j.$$

Then we have

$$I_\rho f = \sum_{j,k} \lambda_j \lambda_{j,k} a_{j,k}.$$

Since $\lambda_j \le 2$, $\lambda_{j,k} \le C$, we have

$$\sum_{j,k} V(\lambda_j \lambda_{j,k}) \le C \sum_{j,k} U(\lambda_j) V(\lambda_{j,k}) \le C' \sum_j U(\lambda_j) \le 2C' \, U\left(\|f\|_{H_U^\Phi}\right).$$

Hence

$$V\left(\|I_\rho f\|_{H_V^\Psi}\right) \le C \, U\left(\|f\|_{H_U^\Phi}\right) \quad \text{for} \quad \|f\|_{H_U^\Phi} \le 1. \quad \square$$

Proposition 7.3. *Under the assumption of Theorem 6.2, if $a \in A(\Phi, \infty)$, then*

$$I_\rho a \in L^\Psi(\mathbb{R}^n), \quad \textit{and} \quad \|I_\rho a\|_{L^\Psi} \le C.$$

where C is independent of $a \in A(\Phi, \infty)$.

Proof of Theorem 6.2. Since $H_U^\Phi \subset H_I^\Phi$, we show $I_\rho : H_I^\Phi \to L^\Psi$. Let

$$f \in L^{\infty,0}_{\text{comp}}(\mathbb{R}^n), \quad f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \le 2\|f\|_{H_I^\Phi}.$$

We can show

$$I_\rho f = \sum_j \lambda_j I_\rho a_j.$$

By Proposition 7.3, we have

$$\|I_\rho f\|_{L^\Psi} \le \sum_j |\lambda_j| \, \|I_\rho a_j\|_{L^\Psi} \le C \sum_j |\lambda_j| \le 2C\|f\|_{H_I^\Phi}. \quad \square$$

8. Atom with vanishing moments up to order N

Definition 8.1. Let $\Phi \in \mathcal{F}$, $1 < q \le +\infty$, $N = 0, 1, 2, \cdots$ and $r^{1/q}\Phi^{-1}(1/r)$ be almost decreasing. A function a on $\mathbb{R}^n$ is called a (Φ, q, N)-atom if there exists a ball B such that

$$\begin{cases} (i) & \operatorname{supp} a \subset B, \\ (ii) & \|a\|_q \le |B|^{1/q}\Phi^{-1}\left(\dfrac{1}{|B|}\right), \\ (iii) & \displaystyle\int a(x) x^\alpha \, dx = 0 \quad \text{for } |\alpha| \le N. \end{cases}$$

For $N = 0$, a $(\Phi, q, 0)$-atom is simply called a (Φ, q)-atom. A function a on $\mathbb{R}^n$ is called a (Φ, q)-block if there exists a ball B such that (i) and (ii) hold. We denote by $A(\Phi, q, N)$ the set of all (Φ, q, N)-atoms. We also denote by $A(\Phi, q, -1)$ the set of all (Φ, q)-blocks.

Definition 8.2. Let $\Phi, U \in \mathcal{F}$ and U be concave. We define a space $H_U^{\Phi,q,N}(\mathbb{R}^n) \subset \mathfrak{D}'$ as follows:

$f \in H_U^{\Phi,q,N}(\mathbb{R}^n)$ if and only if there exist sequances $\{a_j\} \subset A(\Phi, q, N)$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j a_j \text{ in } \mathfrak{D}' \quad \text{and} \quad \sum_j U(\lambda_j) < +\infty. \tag{8.1}$$

In general, the expression (8.1) is not unique. We define

$$\|f\|_{H_U^\Phi} = \inf\left\{ U^{-1}\left(\sum_j U(\lambda_j)\right) : f = \sum_j \lambda_j a_j \text{ in } \mathfrak{D}' \right\},$$

where the infimum is taken over all expressions (8.1). For $\Phi(r) = 1/U(1/r)$, we denote $H_U^{\Phi,q,N}(\mathbb{R}^n) = H^{\Phi,q,N}(\mathbb{R}^n)$.

$H_U^{\Phi,q,N}(\mathbb{R}^n)$ is a linear space. Let $d(f,g) = U(\|f - g\|_{H_U^{\Phi,q,N}})$ for $f, g \in H_U^{\Phi,q,N}(\mathbb{R}^n)$. Then $d(f,g)$ is a metric and $H_U^{\Phi,q,N}(\mathbb{R}^n)$ is a complete with respect to this metric.

If $\Phi(r) = r^p$, $n/(n+N+1) < p \le n/(n+N)$, $N = 0, 1, 2, \cdots$, then $H^{\Phi,q,N}(\mathbb{R}^n) = H^{\Phi,\infty,N}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$.

In the following, we assume that $\Phi, \Psi, U, V \in \mathcal{F}$, that Φ^{-1} and Ψ^{-1} satisfy the doubling condition, that U and V are concave, that $1 < q \le \infty$, $1/q+1/q' = 1$, that $N = -1, 0, 1, 2, \cdots$, that

$$\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty,$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2,$$

$$\frac{\rho(r)}{r^{n+N+1}} \le C\frac{\rho(s)}{s^{n+N+1}} \quad \text{for} \quad s \le r,$$

and that, if $N \ne -1$, then $\rho \in C^N(0, +\infty)$ and

$$\left| \frac{\rho(|u|)}{|u|^n} - \sum_{|\alpha| \le N} Q_\alpha(v)(u-v)^\alpha \right| \le C|u-v|^{N+1}\frac{\rho(|v|)}{|v|^{n+N+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{|u|}{|v|} \le 2,$$

where $Q_\alpha = \frac{1}{\alpha!}\left(\frac{\rho(|\cdot|)}{|\cdot|^n}\right)^{(\alpha)}$.

Theorem 8.1. *Let*

$$\Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right), \quad r > 0,$$

$$V(rs) \le CV(r)U(s), \quad 0 \le r, s \le 1.$$

Let $N' \le N$,

(8.2)
$$\int_1^{+\infty} \frac{\rho(t)}{t^{N-N'+2}}\,dt < +\infty,$$

and $0 < \exists\theta < 1$ s.t.

$$\int_r^{+\infty} V\left(\left(\frac{\Psi^{-1}(1/t^n)}{\Psi^{-1}(1/r^n)}\right)^{(1/\theta)-1}\right) t^{-1}\,dt \le C, \quad r > 0,$$

(8.3)
$$\int_r^{+\infty} t^{n+N'}\left(\Psi^{-1}\left(\frac{1}{t^n}\right)\right)^{1/\theta} t^{-1}\,dt \le Cr^{n+N'}\left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{1/\theta}, \quad r > 0,$$

$$\frac{\rho(r)}{r^{n+N+1}}\left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{-1/\theta} \quad \textit{is almost decreasing.}$$

If $N' = -1$, then we omit (8.2) and (8.3). Then

$$I_\rho : H_U^{\Phi,\infty,N}(\mathbb{R}^n) \to H_V^{\Psi,\infty,N'}(\mathbb{R}^n) \quad \textit{conti.}$$

The proof of Theorem 8.1 is the same as Theorem 6.1. To prove the theorem, we define molecules with vanishing moments up to order N, and state propositions.

Definition 8.3. Let $\Phi \in \mathcal{F}$, $1 < q \leq \infty$, $N = 0, 1, 2, \cdots$, and $0 < \theta < 1$. A function M on $\mathbb{R}^n$ is called a (Φ, q, N, θ)-molecule if

$$\begin{cases} (i) & \exists z \in \mathbb{R}^n \quad \text{s.t.} \quad \|M\|_q^{1-\theta} \left\|b(|\cdot - z|^n)^{1/\theta} M(\cdot)\right\|_\infty^\theta < +\infty, \\ (ii) & \int |M(x)||x|^N \, dx < +\infty, \\ (iii) & \int M(x) x^\alpha \, dx = 0 \quad \text{for } |\alpha| \leq N, \end{cases}$$

where

$$b(r) = \left(r^{1/q} \Phi^{-1}(1/r)\right)^{-1}.$$

A function M on $\mathbb{R}^n$ is called a $(\Phi, q, -1, \theta)$-molecule if (i) halds. Let

$$\mathcal{N}(M) = \mathcal{N}^{\Phi,q,\theta}(M) = \inf_{z \in \mathbb{R}^n} \|M\|_\infty^{1-\theta} \left\|b(|\cdot - z|^n)^{1/\theta} M(\cdot)\right\|_\infty^\theta.$$

Proposition 8.2. *Let $N' \leq N$ and*

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \int_0^r \frac{\rho(t)}{t} \, dt \leq C \Psi^{-1}\left(\frac{1}{r^n}\right), \quad r > 0,$$

$$0 < \exists \theta < 1 \quad s.t. \quad \frac{\rho(r)}{r^{n+N+1}} \left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{-1/\theta} \text{ is almost decreasing}$$

If $N' \neq -1$, we assume that

$$\int_1^{+\infty} \frac{\rho(t)}{t^{N-N'+2}} \, dt < +\infty.$$

If a is a (Φ, ∞, N)-atom, then $I_\rho a$ is a $(\Psi, \infty, N', \theta)$-molecule and $\mathcal{N}(I_\rho a) \leq C$, where C is independent of the (Φ, ∞, N)-atom a.

Proposition 8.3. *Let $0 < \exists \theta < 1$ s.t.*

$$\int_r^\infty V\left(\left(\frac{\Psi^{-1}(1/t^n)}{\Psi^{-1}(1/r^n)}\right)^{(1/\theta)-1}\right) t^{-1} \, dt \leq C, \quad r > 0,$$

$$(8.4) \quad \int_r^{+\infty} t^{n+N} \left(\Psi^{-1}\left(\frac{1}{t^n}\right)\right)^{1/\theta} t^{-1} \, dt \leq C r^{n+N} \left(\Psi^{-1}\left(\frac{1}{r^n}\right)\right)^{1/\theta}, \quad r > 0.$$

If $N = -1$, then we omit (8.4). If M is a (Ψ, q, N, θ)-molecule, then $M \in H_V^{\Psi,q,N}(\mathbb{R}^n)$, and

$$\forall C_1 > 0 \, \exists C_2 > 0 \quad s.t. \quad \mathcal{N}^{\Psi,q,N,\theta}(M) \leq C_1 \Rightarrow \|M\|_{H_V^{\Psi,q,N}} \leq C_2.$$

REFERENCES

[1] J.García-Cuerva and J.L.Rubio de Francia, Weighted norm inequalities and related topics, North-Holland, 1985.

[2] S. Lu, M. H. Taibleson and G. Weiss, Spaces generated by blocks, Publishing House of Beijing Normal University, 1989.

[3] E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math. **5** (2001), 587–602.

[4] ______ *On generalized fractional integrals in the Orlicz spaces on spaces of homogeneous type*, Sci. Math. Jpn. **54** (2001), 473–487. (Sci. Math. Jpn. Online **4** (2001), 901–915).

[5] ______ *On generalized fractional integrals on the weak Orlicz spaces,* BMO_ϕ, *the Morrey spaces and the Campanato spaces*, "Function Spaces, Interpolation Theory and Related Topics: Proceedings of the International Conference in honour of Jaak Peetre on his 65th birthday, Lund, Sweden, August 17–22, 2000 / Editors: Michael Cwikel, Miroslav Engliš, Alois Kufner, Lars-Erik Persson and Gunnar Sparr / Walter de Gruyter, Berlin, New York, 2002", 389–401.

[6] C. T. Zorko, *Morrey space*, Proc. Amer. Math. Soc. **98** (1986), 586–592.

DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, KASHIWARA, OSAKA 582-8582, JAPAN

E-mail address: enakai@cc.osaka-kyoiku.ac.jp